DIFFERENTIAL IVARIANTS. SIMPLEST EXAMPLES. II.

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1. Introduction

In this lecture, we show that a 3-web exists on every solution of a PDE system describing the 1-dimensional gas dynamics. We construct the first nontrivial differential invariant of 3-webs. This invariant is an obstruction for a 3-web to be locally flat. Finally, we calculate explicit solutions of the PDE system possessing locally flat 3-webs.

Below, all manifolds and maps are supposed to be smooth. By $[f]^k_p$, $k = 0, 1, 2, \ldots$, we denote the k-jet of a map f at a point p, by $\mathbb{R}$ we denote the field of real numbers, and by $\mathbb{R}^n$ we denote the n-dimensional arithmetic space.

2. Equations of the 1-dimensional gas dynamics

2.1. Consider the PDE system describing the 1-dimensional gas dynamics

$$\begin{cases} u_t + uu_x + \dfrac{1}{\rho}p_x = 0 \\ \rho_t + u\rho_x + \rho u_x = 0 \\ p_t + up_x + A(\rho, p)u_x = 0\,, \end{cases} \tag{2.1}$$

where u is a velocity, ρ is a density, p is a pressure, and $A(\rho, p) = -\rho(\partial s/\partial\rho)/(\partial s/\partial p)$, where $s(\rho, p)$ is an entropy. We will consider this system as a submanifold in the corresponding jet bundle. To this end consider the following trivial bundle

$$\pi : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2\,, \quad \pi : (x^1, x^2; u^1, u^2, u^3) \mapsto (x^1, x^2)\,.$$

Let $\pi_k : J^k\pi \to \mathbb{R}^2$, $k = 0, 1$, be the bundle of all k-jets of sections of π. By x^j, u^i, u^i_1, u^i_2 we denote the standard coordinates on $J^1\pi$. Obviously, we can consider system (2.1) as a submanifold of $J^1\pi$ defined by the system of equations

$$\begin{cases} u^1_1 + u^1u^1_2 + \dfrac{1}{u^2}u^3_2 = 0 \\ u^2_1 + u^1u^2_2 + u^2u^1_2 = 0 \\ u^3_1 + u^1u^3_2 + A(u^2, u^3)u^1_2 = 0\,, \end{cases} \tag{2.2}$$

We denote this submanifold by $\mathcal{E}$.

Any section S of π generates the section $j_1S : p \mapsto [S]^1_p$ of $J^1\pi$. By definition, put $L^1_S = \operatorname{Im} j_1S$. We identify a solution

$$S : (x^1, x^2) \mapsto \big(u^1 = S^1(x^1, x^2), u^2 = S^2(x^1, x^2), u^3 = S^3(x^1, x^2)\big) \tag{2.3}$$

of system (2.2) with the 2-dimensional submanifold $L^1_S \subset \mathcal{E}$.

Below, by M we denote the base of π.

2.2. **Characteristic covectors.** Recall the coordinate definition of a characteristic covector. Let $\theta_1 \in \mathcal{E}$ and $p = \pi_1(\theta_1)$. A nonzero 1-form $\alpha_1 dx^1 + \alpha_1 dx^2 \in T_p^* M$ is called a *characteristic covector for* θ_1 if

$$\det\left((\partial F^i/\partial u^j_1)\cdot\alpha_1 + (\partial F^i/\partial u^j_2)\cdot\alpha_2 \right)\Big|_{\theta_1} = 0\,,$$

where the functions F^1, F^2, and F^3 are the left hand side of first, second, and third equation of system (2.2) respectively. Explicitly, the last equation is the following

$$\det\begin{pmatrix} \alpha_1 + u^1\alpha_2 & 0 & \alpha_2/u^2 \\ u^2\alpha_2 & \alpha_1 + u^1\alpha_2 & 0 \\ A\alpha_2 & 0 & \alpha_1 + u^1\alpha_2 \end{pmatrix}\Bigg|_{\theta_1} = 0$$

That is

$$(\alpha_1 + u^1\alpha_2)^3 - (\alpha_1 + u^1\alpha_2)A\alpha_2^2/u^2 = 0$$

Obviously, this equation is equivalent to the following two equations

$$\alpha_1 + u^1\alpha_2 = 0\,, \quad \alpha_1 + (u^1 \pm \sqrt{A/u^2})\alpha_2 = 0$$

Note that if ω is a characteristic covector, then $\lambda\,\omega$ is a characteristic covector too for any $\lambda \neq 0$. Taking into account this remark, we obtain that for θ_1 there are three characteristic covectors up to scalar factors:

$$\begin{aligned} \omega^1 &= -u^1 dx^1 + dx^2, \\ \omega^2 &= (-u^1 + \sqrt{A/u^2})dx^1 + dx^2, \\ \omega^3 &= (-u^1 - \sqrt{A/u^2})dx^1 + dx^2 \end{aligned}$$

Obviously, any two of them are linearly independent.

2.3. **3-webs of solutions.** Let L^1_S be an arbitrary solution of $\mathcal{E}$. In standard coordinates it is defined by parametric equations

$$j_1 S : (x^1, x^2) \mapsto \left(u^i = S^i(x^1, x^2), u^i_j = \frac{\partial S^i}{\partial x^j}(x^1, x^2)\right).$$

Substituting S^i for u^i in ω^1, ω^2, and ω^3, we obtain three differential 1-forms

$$\begin{aligned} \omega^1_S &= -S^1 dx^1 + dx^2, \\ \omega^2_S &= (-S^1 + \sqrt{A/S^2})dx^1 + dx^2, \\ \omega^3_S &= (-S^1 - \sqrt{A/S^2})dx^1 + dx^2 \end{aligned} \tag{2.4}$$

on solution S considering as the 2-dimensional manifold L^1_S. Clear, any two of these differential 1-forms are linearly independent at every point of L^1_S.

Let F_i, $i = 1, 2, 3$, be the family of curves on L^1_S so that

$$\forall\, \gamma \in F_i \quad \omega^i_S\big|_\gamma = 0\,.$$

It is easy to check that the following properties hold for these families:

(1) for every i and every point $p \in L^1_S$ there exist a unique $\gamma_i \in F_i$ passing trough p.
(2) for $i \neq j$, any two curves $\gamma_i \in F_i$ and $\gamma_j \in F_j$ intersect transversally.

The first property follows immediately from the theory of ordinary differential equations. Second one follows immediately from the linear independence of any two of the forms ω^1, ω^2, and ω^3 at every point $p \in L^1_S$.

Definition 2.1. *A collection of three families of curves $W = \{F_1, F_2, F_3\}$ on a 2-dimensional manifold is called a 3-web if these families satisfy conditions* (1) *and* (2).

Thus, 1-forms (2.4) define the 3-web on the solution L^1_S of $\mathcal{E}$. By W_S we denote this 3-web.

3. Differential invariants of 3-webs

Let L be a smooth 2-dimensional manifold, $W = \{F_1, F_2, F_3\}$ a 3-web on L, and $f : L \to L$ a diffeomorphism. Then f transforms any curve $\gamma_i \in F_i$, $i = 1, 2, 3$, to the curve $f(\gamma_i)$. Obviously, these transformed curves define the new 3-web. This 3-web is called a *transformed 3-web* and it is denoted by $f(W)$.

A function or a differential form Ω_W generated by W by some rule is called a *differential invariant of* W if for any diffeomorphism f the following condition is satisfied

$$\Omega_W = f^*(\Omega_{f(W)}),$$

where $\Omega_{f(W)}$ is generated by $f(W)$ by the same rule.

Let W' be another 3-web on L. The 3-webs W and W' are called *locally equivalent* if there exist a local diffeomorphism

$$L \supset U \xrightarrow{f} U' \subset L$$

such that $f(W|_U) = W'|_{U'}$. The problem to find necessary and sufficient conditions of existence of a local diffeomorphism transforming one 3-web to another one is called the *equivalence problem.* A complete collection of differential invariants make possible to solve this problem.

Now we construct the first nontrivial differential invariant of 3-webs.

It is clear that a collection of differential 1-forms generating a 3-web is not uniquely defined. In fact, if the collection of 1-forms α^1, α^2, and α^3 defines W, then for everywhere nonzero functions f_1, f_2, f_3, the collection of 1-forms $f_1\alpha^1$, $f_2\alpha^2$, $f_3\alpha^3$ defines W too. It follows that we can choose forms α^1, α^2, and α^3 defining W so that

$$\alpha^1 + \alpha^2 + \alpha^3 = 0\,. \tag{3.1}$$

These 1-forms are defined uniquely up to a common everywhere nonzero factor. Let

$$\Theta = \alpha^1 \wedge \alpha^2\,.$$

From (3.1), we have $\alpha^1 \wedge \alpha^2 = \alpha^2 \wedge \alpha^3 = \alpha^3 \wedge \alpha^1$. From $\dim L = 2$, it follows that

$$d\alpha^i = \lambda_i \Theta, \quad i = 1, 2, 3\,.$$

Let

$$\sigma = \lambda_1\alpha^2 - \lambda_2\alpha^1\,.$$

Then from (3.1), we obtain $\lambda_1\alpha^2 - \lambda_2\alpha^1 = \lambda_2\alpha^3 - \lambda_3\alpha^2 = \lambda_3\alpha^1 - \lambda_1\alpha^3$. By definition, put

$$\Omega_W = d\sigma\,.$$

It is easy to prove the following statement.

Proposition 3.1. *The 2–form* Ω_W *is independent of the choice of* α^1, α^2, *and* α^3 *satisfying* (3.1).

The 2–form α is called the *curvature form of* W. It is a differential invariant of W.

A 3-web on L is called *locally flat* if for any $p \in L$ there exist a local chart in a neighborhood of p such that curves of W expressed in terms of this chart are straight lines.

Theorem 3.2. *A 3-web* W *is locally flat iff* $\Omega_W = 0$.

4. Explicit solutions possessing locally flat 3-webs

4.1. Following the previous section, let us calculate the curvature of the 3-web W_S defined on the solution L^1_S of system (2.1). This web is defined by 1-forms (2.4)

$$\omega^1_S = -udt + dx, \;\; \omega^2_S = (-u + \sqrt{A/\rho})dt + dx, \;\; \omega^3_S = (-u - \sqrt{A/\rho})dt + dx$$

Putting $\alpha^1 = -2\omega^1_S$, $\alpha^2 = \omega^2_S$, and $\alpha^3 = \omega^3_S$, we obtain $\alpha^1 + \alpha^2 + \alpha^3 = 0$,

$$\begin{aligned}
\Theta &= \alpha^1 \wedge \alpha^2 = 2\sqrt{A/\rho} \cdot dt \wedge dx \\
d\alpha^1 &= \lambda_1 \Theta = -\frac{u_x}{\sqrt{A/\rho}} \Theta\,, \\
d\alpha^2 &= \lambda_2 \Theta = \frac{(u - \sqrt{A/\rho})_x}{2\sqrt{A/\rho}}\, \Theta\,, \\
\sigma &= \lambda_1 \alpha^2 - \lambda_2 \alpha^1 = \big(u(\ln|\sqrt{A/\rho}|)_x - u_x \big) dt - (\ln|\sqrt{A/\rho}|)_x \, dx\,,
\end{aligned}$$

and finally

$$\begin{aligned}
\Omega_{W_S} = d\sigma = \frac{\rho}{A}\Big(&(A/\rho) u_{xx} - \sqrt{A/\rho}(\sqrt{A/\rho})_x u_x \\
+ \big((A/\rho)_x - \sqrt{A/\rho}(\sqrt{A/\rho})_{xx}\big) u &- (\sqrt{A/\rho})_{tx} + (\sqrt{A/\rho})_t (\sqrt{A/\rho})_x \Big) dt \wedge dx\,.
\end{aligned} \tag{4.1}$$

4.2. From theorem 3.2 and equation (4.1), it follows that the 3-web W_S is locally flat iff the solution S satisfies additionally the equation

$$\begin{aligned}
(A/\rho) u_{xx} - \sqrt{A/\rho}(\sqrt{A/\rho})_x u_x + \big((A/\rho)_x - \sqrt{A/\rho}(\sqrt{A/\rho})_{xx}\big) u \\
- (\sqrt{A/\rho})_{tx} + (\sqrt{A/\rho})_t (\sqrt{A/\rho})_x = 0
\end{aligned} \tag{4.2}$$

For a simplicity, consider a special case of system (2.1):

$$A(\rho, p) = \rho\,.$$

Then equation (4.2) is

$$u_{xx} = 0\,.$$

Let us find explicit solutions of this special system possessing locally flat 3-webs. To this end, we should solve the system

$$u_t + uu_x + \rho_x/\rho = 0 \tag{4.3}$$

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{4.4}$$

$$p_t + up_x + \rho u_x = 0 \tag{4.5}$$

$$u_{xx} = 0\,. \tag{4.6}$$

From (4.6), we have

$$u = c_1(t)x + c_2(t)\,. \tag{4.7}$$

Substitute (4.7) in (4.4). Solving obtained equation, we get a general solution

$$\rho = e^{-\int c_1}\varphi\big(xe^{-\int c_1} - \int c_2 e^{-\int c_1}\big)\,,$$

where φ is an arbitrary smooth function. Putting $\varphi \equiv 1$, we obtain a special solution

$$\rho = e^{-\int c_1}\,. \tag{4.8}$$

Substitute (4.7) and (4.8) into (4.5). Solving obtained equation, we get a general solution

$$p = -\int c_1 e^{-\int c_1} + \psi\big(xe^{-\int c_1} - \int c_2 e^{-\int c_1}\big)\,,$$

where ψ is an arbitrary smooth function. Putting $\psi \equiv \mathrm{id}$, we obtain a special solution

$$p = xe^{-\int c_1} - \int e^{-\int c_1}(c_1 + c_2)\,. \tag{4.9}$$

Finally, substituting (4.7), (4.8), and (4.9) into (4.3), we obtain

$$c_1 = \frac{1}{t + K_1}\,, \quad c_2 = -\frac{1}{2}(t + K_1) + \frac{K_2}{t + K_1}\,,$$

where $K_1, K_2 \in \mathbb{R}$. Substituting these expressions for c_1 and c_2 into (4.7), (4.8), and (4.9), we obtain the following solutions of system (2.1) possessing the locally flat 3-webs:

$$u = \frac{x + K_2}{t + K_1} - \frac{t + K_1}{2}$$

$$\rho = \frac{K_3}{|t + K_1|}$$

$$p = \Big[\frac{x}{|t + K_1|} - (1 + K_2)\Big(K_4 - \frac{1}{|t + K_1|}\Big) + \frac{1}{2}(|t + K_1| + K_5)\Big]K_3\,,$$

here $K_1, \dots, K_5 \in \mathbb{R}$.

5. Exercises

1. (1) Let $W = \{F_1, F_2, F_3\}$ be 3-web on a smooth manifold L. Prove that for any $p \in L$ there exist a local chart in a neighborhood of p such that F^1 and F^2 coincide respectively with the family of first coordinate lines and the family of second coordinate lines of this chart.
2. (2) Prove proposition 3.1.

(3) Prove that the curvature of a 3-web is a differential invariant.
(4) Express the curvature of a 3-web in terms of the chart described in exercise 1.
(5) Prove that locally flat 3-webs are locally equivalent.
(6) Prove theorem 3.2.
(7) Let ξ be a vector field on L and f_t its flow. Then ξ is called a *symmetry of 3-web* W, if $f_t(W) = W$ for every t.
   (a) Prove that the set of all symmetries of W is a Lie algebra.
   (b) Calculate the symmetry algebra for an arbitrary 3-web.
(8) Let $A(\rho, p) = \rho$ in system (2.1). Find solutions of this system possessing locally flat 3-webs.

References

[1] W. Blaschke, *Introduction to the Gemetry of Webs* [Russian translation], Fizmatgiz, Moscow, 1959.
[2] A.M. Vinogradov, V.A. Yumaguzhin, *Differential invariants of webs on 2-dimensional manifolds*, Mat. Zametki Vol. 48, No. 1, pp. 46 - 68, 1990 (in Russian, English translationin: Soviet Mat. Zametki).
[3] E.V. Ferapontov, *Hydrodynamic type equations from the web theory point of view*, Mat. Zametki Vol. 50, No. 5, pp. 97 - 108, 1991 (in Russian, English translationin: Soviet Mat. Zametki)

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